KUWAIT UNIVERSITY

Department of Mathematics & Computer Science

Math 102 | | Aug 2, 2008

Calculus II | Final Exam | Time: 120 minutes

No calculators or cell phones are permitted during the exam

1. Given that $f(x) = \sqrt{x^3 - x^2 + x}$. [4 pts]

   (a) Show that f is one-to-one in the interval $[0, \infty)$.

   (b) Find $(f^{-1})'(1)$.

2. Find the derivative of $y = \tan^{-1}(x\,e^{-x}) + \ln\left[(3^x + 1)\,(2 + \cos x)\right]$. [4 pts]

3. Evaluate $\displaystyle\int \frac{1}{\cosh x - 1}\,dx$. [3 pts]

4. Evaluate $\displaystyle\int \frac{1}{x^5 + x}\,dx$. [3 pts]

5. Find $\displaystyle\lim_{x\to\infty} \left(\frac{x+2}{x-1}\right)^x$. [3 pts]

6. Evaluate the improper integral $\displaystyle\int_0^\infty \frac{x}{e^x}\,dx$ if it converges. [3 pts]

7. The curve $y = \dfrac{x^2}{2} - \dfrac{\ln x}{4}$, $2 \le x \le 4$, is rotated about the x-axis. Find the area of the resulting surface. [4 pts]

8. Find the centroid of the region bounded by the curves $y = x^3$, $x + y = 2$, and $x = 0$. [4 pts]

9. Sketch the graphs of the polar equations $r = 3 - 3\sin\theta$ and $r = 1 + \sin\theta$, and find the area of the region that lies inside both graphs. [6 pts]

10. A curve C is given by the parametric equations $x = \sin t + \cos t$ and $y = \sin t - \cos t$, where $-\pi \le t \le \pi$. [6 pts]

    (a) Find the points $P(x, y)$ on C where it has horizontal or vertical tangent lines.

    (b) Find the length of the curve C.

1. Given that $f(x) = \sqrt{x^3 - x^2 + x}$. [4 pts]

   (a) Show that f is one-to-one in the interval $[0, \infty)$.

$$f'(x) = \frac{3x^2 - 2x + 1}{2\sqrt{x^3 - x^2 + x}} = \frac{3}{2}\,\frac{(x-1/3)^2 + 2/9}{\sqrt{x^3 - x^2 + x}} > 0\,, \qquad x > 0$$

   (b) Find $(f^{-1})'(1)$. By inspection, $f(1) = 1$ so that $f'(1) = 1$ and

$$(f^{-1})'(1) = \frac{1}{f'(1)} = 1$$

2. Find the derivative of $y = \tan^{-1}(x\,e^{-x}) + \ln\left[(3^x + 1)\,(2 + \cos x)\right]$. [4 pts]

$$y = \tan^{-1}(x\,e^{-x}) + \ln(3^x + 1) + \ln(2 + \cos x) \;\rightarrow\; y' = \frac{e^{-x} - x\,e^{-x}}{1 + x^2\,e^{-2x}} + \frac{3^x \ln 3}{3^x + 1} - \frac{\sin x}{2 + \cos x}$$

3. Evaluate $\displaystyle\int \frac{1}{\cosh x - 1}\,dx$. [3 pts]

$$\int \frac{\cosh x + 1}{\cosh^2 x - 1}\,dx = \int \frac{\cosh x + 1}{\sinh^2 x}\,dx = \int (\coth x \operatorname{csch} x + \operatorname{csch}^2 x)\,dx = -(\operatorname{csch} x + \coth x) + c$$

OR

$$\int \frac{1}{\cosh x - 1}\,dx = \int \frac{2}{e^x + e^{-x} - 2}\,dx = \int \frac{2\,e^x}{e^{2x} + 1 - 2\,e^x}\,dx = \int \frac{2\,e^x}{(e^x - 1)^2}\,dx = \frac{-2}{e^x - 1} + c$$

4. Evaluate $\displaystyle\int \frac{1}{x^5 + x}\,dx$. Put $u = x^4$ [3 pts]

$$\int \frac{1}{x\,(x^4+1)}\,dx = \frac{1}{4}\int \frac{4x^3}{x^4\,(x^4+1)}\,dx = \frac{1}{4}\int \frac{du}{u\,(u+1)} = \frac{1}{4}\int \left[\frac{1}{u} - \frac{1}{u+1}\right] du = \frac{1}{4}\ln\frac{u}{u+1} + c$$

5. Find $\displaystyle\lim_{x\to\infty}\left(\frac{x+2}{x-1}\right)^x$. $y = \left(\frac{x+2}{x-1}\right)^x$ [3 pts]

$$\ln y = x\,(\ln(x+2) - \ln(x-1)) = \frac{\ln(x+2) - \ln(x-1)}{1/x} \;\rightarrow\; \lim \ln y = \lim \frac{1/(x+2) - 1/(x-1)}{-x^{-2}}$$

$$\lim \ln y = \lim \frac{3x^2}{(x+2)\,(x-1)} = 3 \;\rightarrow\; \lim y = e^3$$

6. Evaluate the improper integral $\displaystyle\int_0^\infty \frac{x}{e^x}\,dx$ if it converges. Integrate by parts: [3 pts]

$$\lim_{t\to\infty}\int_0^t x\,e^{-x}\,dx = \lim_{t\to\infty} -e^{-x}\,(x+1)\Big|_0^t = \lim_{t\to\infty} -\frac{t+1}{e^t} + 1 = 1$$

7. The curve $y = \dfrac{x^2}{2} - \dfrac{\ln x}{4}$, $2 \le x \le 4$, is rotated about the x-axis. Find the area of the resulting surface. [4 pts]

$$y' = x - \frac{1}{4x} \;\to\; (y')^2 + 1 = x^2 - \frac{1}{2} + \frac{1}{16x^2} + 1 = x^2 + \frac{1}{2} + \frac{1}{16x^2} = \left(x + \frac{1}{4x}\right)^2$$

$$S = \int_2^4 2\pi\, y\,\sqrt{1+(y')^2}\,dx 2\pi \int_2^4 \left(\frac{x^2}{2} - \frac{\ln x}{4}\right)\left(x + \frac{1}{4x}\right) dx = 2\pi \int_2^4 \left(\frac{x^3}{2} + \frac{x}{8} - \frac{x\ln x}{4} - \frac{\ln x}{16x}\right) dx$$

$$S = \frac{\pi}{4}\left[x^4 + x^2 - x^2 \ln x - \frac{(\ln x)^2}{4}\right]_2^4 = \cdots$$

8. Find the centroid of the region bounded by the curves $y = x^3$, $x + y = 2$, and $x = 0$. [4 pts]
Intersection point: $x^3 = 2 - x$ gives $x = 1$. The line $y + x = 2$ lies above the curve $y = x^3$ in the first quadrant between the lines $x = 0$ and $x = 1$. The area of the region is

$$A = \int_0^1 (2 - x - x^3)\,dx = \frac{5}{4}$$

and the moments are:

$$M_x = \rho \int_0^1 \frac{1}{2}\left((2-x)^2 - x^5\right) dx = \frac{23}{21}\,\rho, \quad M_y = \rho \int_0^1 x\,(2 - x - x^3)\,dx = \frac{7}{15}\,\rho$$

which give the centroid:

$$\bar{x} = \frac{M_y}{\rho A} = \frac{28}{75}, \quad \bar{y} = \frac{M_x}{\rho A} = \frac{92}{105}$$

9. Sketch the graphs of the polar equations $r = 3 - 3\sin\theta$ and $r = 1 + \sin\theta$, and find the area of the region that lies inside both graphs. [6 pts]
Due to symmetry, we need one intersection point in the first quadrant: $1 + \sin\theta = 3 - 3\sin\theta$ which gives $\sin\theta = 1/2$, so that $\theta = \pi/6$. The area becomes

$$A = 2\left[\int_{-\pi/2}^{\pi/6} \frac{1}{2}(1+\sin\theta)^2\,d\theta + \int_{\pi/6}^{\pi/2} \frac{1}{2}(3 - 3\sin\theta)^2\,d\theta\right] = \cdots = \frac{11}{2}\pi - 9\sqrt{3}$$

10. A curve C is given by the parametric equations $x = \sin t + \cos t$ and $y = \sin t - \cos t$, where $-\pi \le t \le \pi$. [6 pts]

(a) Find the points $P(x, y)$ on C where it has horizontal or vertical tangent lines.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t + \sin t}{\cos t - \sin t} = \frac{1 + \tan t}{1 - \tan t}$$

Horizontal tangent line: $1 + \tan t = 0$ occurs at $t = -\pi/4, 3\pi/4$, that is at $P(0, \mp\sqrt{2})$.
Vertical tangent line: $1 - \tan t = 0$ occurs at $t = \pi/4, -3\pi/4$, that is at $P(\pm\sqrt{2}, 0)$.

(b) Find the length of the curve C.

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (\cos t - \sin t)^2 + (\cos t + \sin t)^2 = 2$$

$$L = \int_{-\pi}^{\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_{-\pi}^{\pi} \sqrt{2}\,dt = 2\sqrt{2}\,\pi$$